TA: LEE, Yat Long Luca
Email: yllee@math.cuhk.edu.hk
Office: Room 505, AB1
Office Hour: Send me an email first, then we will arrange a meeting (if you need it).

Solution will be uploaded after the tutorial on Wednesday.

Recall:

Theorem 3.10 (Picard-Lindelöf Theorem)
Consider the IVP given by

$$\begin{cases} \dfrac{dx}{dt} = f(t,x) \\ x(t_0) = x_0 \end{cases}$$

where $f \in C(R)$ satisfies the Lipschitz condition on

$$R := [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b] = \overline{B_a(t_0)} \times \overline{B_b(x_0)}.$$

Then there exists $a' \in (0,a)$ and $x \in C^1(\overline{B_{a'}(t_0)})$, $x(t) \in \overline{B_b(x_0)}$ for all $t \in \overline{B_{a'}(t_0)}$ such that it solves the IVP above and the solution is unique in $\overline{B_{a'}(t_0)}$.

Recall from your ODE course that ODE of any order can be written as a system of first order ODEs, and that the Picard-Lindelöf theorem is still valid for system of first order IVPs.

Let (X,d) be a metric space

- $C_b(X) \subset C(X)$
- If G is bounded and open in $\mathbb{R}^n$, then $C_b(\overline{G}) = C(\overline{G})$
- $(C_b(X), d_\infty)$ is a complete metric space, for any metric space (X,d)
- A subset $E \subset X$ is **precompact** if every sequence in E contains a convergent subsequence (its limit may or may not be in E). If the limit is in E, then E is **compact**.
- A subset $\mathcal{C}$ of $C(X)$ is **equicontinuous** if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

  $$|f(x) - f(y)| < \varepsilon, \quad \forall f \in \mathcal{C} \text{ and } d(x,y) < \delta$$

  for $x, y \in X$. Any subset of $\mathcal{C}$ is equicontinuous.
- $f : \overline{G} \to \mathbb{R}$ is **Hölder continuous** with exponent $\alpha \in (0,1)$ if

  $$|f(x) - f(y)| \leq L|x-y|^\alpha, \quad \forall x, y \in \overline{G} \text{ and some constant } L$$

Proposition 4.1 Let $\mathcal{C}$ be a subset of $C(\overline{G})$, where $\overline{G}$ is convex in $\mathbb{R}^n$. Suppose that each function in $\mathcal{C}$ is differentiable and there is a uniform bound on their partial derivatives. Then $\mathcal{C}$ is equicontinuous.

More on Compact sets

Definition 1.1 Let (X, d) be a metric space, a set $A \subset X$ is said to be *totally bounded* if for every $\varepsilon > 0$, A can be covered by finitely many open balls of radius ε. Such an open cover by finitely many open balls with radius $\varepsilon > 0$ is called a finite *ε-net*.

Definition 1.2 Let (X, d) be a metric space. A subset $K \subset X$ is said to be *compact* if for any open coverings $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of K, there exists a finite subcovering $\{U_{\alpha_i}\}_{i=1}^n$ of K. That is, if for any $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ such that $K \subseteq \bigcup_{\alpha \in \mathcal{A}} U_\alpha$, then there exists $\{U_{\alpha_i}\}_{i=1}^n \subset \{U_\alpha\}_{\alpha \in \mathcal{A}}$ such that $K \subseteq \bigcup_{i=1}^n U_{\alpha_i}$.

Remark: The "Compactness" defined in MATH3060 is in fact called "sequential compactness". We will show that these two definitions are equivalent over any metric spaces.

Theorem 1.3 Let (X, d) be a metric space and $K \subset X$. Then the following are equivalent:

(i) K is compact

(ii) Every sequence in K has a convergent subsequence which converges in K

(iii) K is complete and totally bounded

Proof:
(i) $\Longrightarrow$ (ii)

We prove it by a contrapositive argument. Suppose that K is not sequentially compact, then there exists a sequence $\{x_n\}$ such that it does not contain a subsequence which converges in K.

For all $x \in K$, if every ball centered at x contains infinitely many elements, then we are done, in the sense that we can construct a converging sequence. We do it by constructing the sequence as follows: Consider the ball $B(x, \frac{1}{k})$ for $k \geq 1$. For $k = 1$, since $B(x, 1)$ contains infinitely many elements, we can pick $x_{n_1} \in B(x, 1)$. Then for $k = 2$, we pick $x_{n_2} \in B(x, \frac{1}{2})$ with $n_2 > n_3$, such a x_{n_2} exists because $B(x, \frac{1}{2})$ contains infinitely many elements. We keep doing it for all k and obtain $\{x_{n_k}\}_k$ with $n_{k+1} > n_k$ for all $k \geq 1$. Then this sequence tends to x as $k \to \infty$. Which contradicts the fact that K is not sequentially compact.

Hence, we deduced that the open balls contain only finitely many elements. Now, for all $x \in K$, pick a ball, B_x which centers at x. Then $\{B_x\}_{x \in K}$ forms an open cover of K. But then a finite subcover of this collection will only cover finitely many points in K, which implies K does not admit a finite subcovering, further implies that K is not compact.

(ii) $\Longrightarrow$ (iii)

Suppose that K is sequentially compact, we would like to show that K is both complete and totally bounded.

We first show that K is complete. Pick a Cauchy sequence $\{x_n\}$ in K, since K is sequentially compact, there exists a subsequence of $\{x_n\}$ such that it converges to a point in K. But this is not enough, we need to show that $\{x_n\}$ converges in K too.

Lemma 1.4 If $\{x_n\}$ is a Cauchy sequence and suppose that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to x$ as $j \to \infty$. Then $x_n \to x$ as $n \to \infty$. *This concludes the part where K is complete, because Cauchy sequence converges to the limit of its subsequence.*

Proof of Lemma 1.4:
We want to show that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that whenever $n > N$, we have $d(x_n, x) < \varepsilon$.

Since we are given that K contains a convergence subsequence, then there exists $J \in \mathbb{N}$ such that for all $j > J$, we have $d(x_{n_j}, x) < \varepsilon/2$. Moreover, since $\{x_n\}$ is Cauchy, there exists $N \in \mathbb{N}$ such that for $m, n > M$, we have $d(x_m, x_n) < \varepsilon/2$.

Since n_j is increasing , there must exists $j > J$ such that $n_j > N$. Then for any $n > N$, we have

$$d(x_n, x) < d(x_n, x_{n_j}) + d(x_{n_j}, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Thus, we conclude Lemma 1.4 and completeness of K.

□

Now we proceed to show that K is totally bounded. That is, we want to show that for every $\varepsilon > 0$, K can be covered by finitely many open balls with radius ε. We show this by constructing a finite ε-net manually.

Pick $x_1 \in K$, if $B(x_1, \varepsilon)$ covers K, then we are done. If not, since $B(x_1, \varepsilon)$ does not cover K, we can pick $x_2 \in K \setminus B(x_1, \varepsilon)$, if $B(x_1, \varepsilon) \cup B(x_2, \varepsilon)$ covers K, then we are done. We can repeat this process for any x_n. S

Now, suppose the above process ends after finitely many steps, then we are done. If not, i.e., we can pick a point indefinitely, then we get an infinite sequence $\{x_n\}_{n\in\mathbb{N}}$ such that each x_n does not lie in $B(x_1\varepsilon) \cup \cdots \cup B(x_{n-1}, \varepsilon)$. In particular, $d(x_n, x_m) \geq \varepsilon$ for all $n \neq m$. This implies the sequence does not have a Cauchy subsequence. Since all convergent sequence is Cauchy, this implies $\{x_n\}$ has no convergent subsequence, which contradicted the fact that K is sequentially compact. Thus this process ends with finitely many steps, hence obtaining a finite ε-net.

(iii) $\Longrightarrow$ (i)
(Leave it for the next tutorial. Will continue once you have learnt the diagonalization argument from the proof of Arzelà-Ascoli's theorem.)

■

Exercise 1

Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : X \to Y$ be a continuous map. Show that if $K \subseteq X$ is compact in X, then $f(K)$ is compact in Y.

Solution:
Since we want to show that $f(K)$ is compact, we may pick any open covering $\{U_\alpha\}_{\alpha \in I}$ of $f(K)$ such that

$$f(K) \subseteq \bigcup_{\alpha \in I} U_\alpha$$

then show that there is a finite subcovering.

Since K is compact, and that $\{U_\alpha\}_{\alpha \in I}$ is an open covering of $f(K)$, then $\{f^{-1}(U_\alpha)\}_{\alpha \in I}$ is an open covering of K, since

$$f(K) \subseteq \bigcup_{\alpha \in I} U_\alpha \implies K \subseteq f^{-1}\left(\bigcup_{\alpha \in I} U_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(U_\alpha)$$

then using the fact that K is compact, there exists a finite subcollection of $\{f^{-1}(U_\alpha)\}_{\alpha \in I}$ such that

$$K \subseteq \bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i})$$

similarly,

$$f(K) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$$

thus, $\{U_{\alpha_i}\}_{i=1}^{n}$ is a finite subcovering of K.

■

Exercise 2: Grönwall's Inequality (Basic Case)

This a useful inequality in the theory of ODE.

Let $L > 0$ be a positive constant, and C be any real constant. Suppose $y(t)$ is a continuous function defined on a time interval I containing t_0 and satisfies

$$y(t) \leq C + \int_{t_0}^{t} Ly(s)\,ds \tag{0.1}$$

for all $t \in I$. Then we have $y(t) \leq Ce^{L|t-t_0|}$ for all $t \in I$.

Prove it.

Solution:
In this proof, we will show the case for $t > t_0$. The proof for $t < t_0$ is done similarly, thus it is left for you to verify.

The proof of the Grönwall's inequality uses a common technique called *barrier method*. The idea of the barrier method is to see whether the solution, $x(t) = Ce^{L|t-t_0|}$, of the integral equation

$$x(t) = C + \int_{t_0}^{t} Lx(s)\,ds \tag{0.2}$$

lies above the $y(t)$ satisfying (0.1). Graphically, the desired result would be $y(t) \leq x(t)$ for all $t \geq t_0$ where $t \in I$.

We will apply a commonly used trick in theory of ODE/PDE, called "ε-*trick*", to show our desired result.

Given any $\varepsilon > 0$, for any $t \in I$, equation (0.1) can be written as

$$y(t) < (C + \varepsilon) + \int_{t_0}^{t} Ly(s)\,ds \tag{0.3}$$

Let $x_\varepsilon(t) := (C + \varepsilon)e^{L|t-t_0|}$, which satisfies

$$x_\varepsilon(t) = (C + \varepsilon) + \int_{t_0}^{t} Lx_\varepsilon(s)\,ds \tag{0.4}$$

for any $t \in I$.

At the point $t = t_0$, from (0.3) and (0.4), we see that $y(t_0) < C + \varepsilon$ and $x_\varepsilon(t_0) = C + \varepsilon$, meaning that $y(t_0) < x_\varepsilon(t_0)$.

Since we want to show that $y(t) < x_\varepsilon(t)$ for all $t \geq t_0$ and $t \in I$, we assume that there exists a $t_1 > t_0$ such that $y(t_1) = x_\varepsilon(t_1)$, such a t_1 is chosen such that this is the **first** time y and x_ε intersects. In other words, $y(t) < x_\varepsilon(t)$ for all $t \in [t_0, t_1)$. Then

$$\int_{t_0}^{t_1} (x_\varepsilon(s) - y(s))\,ds > 0 \tag{0.5}$$

However, if we substitute t_1 into (0.3) and (0.4), then

$$0 = y(t_1) - x_\varepsilon(t_1) < \int_{t_0}^{t_1} L(y(s) - x_\varepsilon(s))\, ds$$

but (0.5) tells us that the integral should be strictly larger than 0. This, a contradiction.

Hence, $y(t) < x_\varepsilon(t)$ for all $t \geq t_0$. Since ε is chosen arbitrarily, we may take $\varepsilon \to 0^+$ and thus

$$y(t) \leq \lim_{\varepsilon \to 0^+} (C + \varepsilon) e^{L(t-t_0)} = Ce^{L(t-t_0)}$$

for any $t \geq t_0$ and $t \in I$.

■

Exercise 3: Grönwall's Inequality (Variation)

This version of the Grönwall's inequality replaced the positive constant L in the basic case by a nonnegative continuous function $v : (-\infty, \infty) \to \mathbb{R}$.

Let C be any real constant and $v : (-\infty, \infty) \to \mathbb{R}$ be a nonnegative continuous function. Suppose $u : [0, \alpha] \to \mathbb{R}$ is a continuous function such that

$$u(t) \leq C + \int_0^t v(s)u(s)\, ds \tag{0.6}$$

for all $t \in [0, \alpha]$. Then

$$u(t) \leq C \exp\left(\int_0^t v(s)\, ds\right) \tag{0.7}$$

for all $t \in [0, \alpha]$.

Prove it.

Solution:
For any $\varepsilon > 0$, from (0.6), we know that

$$u(t) < (C + \varepsilon) + \int_0^t v(s)u(s)\, ds \tag{0.8}$$

Now consider

$$x_\varepsilon(t) = (C + \varepsilon) \exp\left(\int_0^t v(s)\, ds\right)$$

we verify that it is a solution to the integral equation

$$f(t) = (C + \varepsilon) + \int_0^t v(s)f(s)\, dt \iff f'(t) = v(t)f(t)$$

(f is used for generality). Differentiate $x_\varepsilon(t)$, we have

$$\begin{aligned}
x'_\varepsilon(t) &= (C + \varepsilon)\frac{d}{dt} \exp\left(\int_0^t v(s)\, ds\right) \\
&= (C + \varepsilon) \exp\left(\int_0^t v(s)\, ds\right) \frac{d}{dt}\left(\int_0^t v(s)\, ds\right) \\
&= (C + \varepsilon) \exp\left(\int_0^t v(s)\, ds\right) v(t) \\
&= v(t)x_\varepsilon(t)
\end{aligned}$$

thus it is a solution. For later purposes, we write

$$x_\varepsilon(t) = (C + \varepsilon) + \int_0^t v(s)x_\varepsilon(s)\, ds \tag{0.9}$$

Now we take a time $t_1 \in [0, \alpha]$ such that $t_1 > t_0$ and $u(t_1) = x_\varepsilon(t_1)$, which is the first time such that u intersects x_ε . Then

$$\int_0^{t_1} (u(s) - x_\varepsilon(s))\, ds > 0$$

since v is nonnegative continuous,

$$\int_0^{t_1} v(s)(u(s) - x_\varepsilon(s))\,ds \geq 0 \tag{0.10}$$

However, by (0.8) and (0.9) and substituting $t = t_1$, we have

$$0 = u(t_1) - x_\varepsilon(t_1) < \int_0^t v(s)(u(s) - x_\varepsilon(s))\,ds$$

which contradiction (0.10). Thus, $u(t) < x_\varepsilon(t)$ for all $t \in [0, \alpha]$. Taking $\varepsilon \to 0^+$, we have

$$u(t) \leq \lim_{\varepsilon \to 0^+} (C + \varepsilon) \exp\left(\int_0^t v(s)\,ds\right) = C \exp\left(\int_0^t v(s)\,ds\right)$$

■

Reference

1. Analysis II by T. Tao
2. MATH3060 Lecture notes by Prof KS Chou.
3. Ordinary Differential Equation by Prof Frederick TH Fong
4. Principle of Mathematical Analysis by W. Rudin